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Geometry and analysis on isospectral sets. II. Grassmannian, determinant bundle and the tau function, asymptotic case

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Abstract

First the definition of the infinite dimensional Grassmannian over the Hilbert space $L^2_{\mathbb{R}}[0, 1] = E \oplus U$ is given, where the polarization is according to the even and odd functions. We then prove that the normal space $N_q(M(p))$ at the point $q \in M(e)$, where $M(e)$ is the isospectral set to the even point e, belongs to the Grassmannian over $L^2_{\mathbb{R}}[0, 1]$. Since an element of the normal space depends not only on its base point but also on a parameter $x \in [0, 1]$, we first fix x, and for arbitrary $q \in M(e)$ we derive the group action on the Grassmannian. The determinant bundle and the tau function are constructed over the Grassmannian whereas in contrast to the usual constructions regularized determinants are involved. For fixed elements of the isospectral set and the variation of the parameter x the second group which acts on the Grassmannian is derived. It is then shown how a unified construction for both group acting on the Grassmannian has to be carried out by comparing it to the Fock bundle construction of 3 + 1 dimensional Dirac–Yang–Mills theories.

Keywords: Isospectral sets; Grassmannians; Determinant bundle; Tau function; Asymptotic case
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0. Introduction

In this paper we first introduce the notion of the Grassmannian over an infinite dimensional Hilbert space. The Grassmannian turns out to be a Hilbert manifold modeled over the space of the Hilbert–Schmidt operators and there exists a group which acts on it. The difficulties in our case arise from our Hilbert space $L^2_{\mathbb{R}}[0, 1]$ in contrast to the space $L^2(S^1, \mathbb{C})$ studied

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in [4,10,17]. This complicates the matter essentially since in their case the operators act as multiplication operators whereas in our case the way the group acts on the Grassmannian has to be recovered in a more difficult way. Therefore, we discuss in this paper the toy model of asymptotic eigenvalues which considerably simplifies the analysis. We first prove that the normal spaces at any point $q \in L^2_{\mathbb{R}}[0,1]$ of the isospectral set $M(p)$ belong to the Grassmannian of $L^2_{\mathbb{R}}[0,1]$. The group acting on the Grassmannian is found by using the analytical facts from [11]. We then construct the determinant bundle and the tau function for the Grassmannian and determine the group acting on it for fixed $x \in [0,1]$. In the next section the group acting on the normal vectors which keeps the points $q \in L^2_{\mathbb{R}}[0,1]$ fixed but vary the parameter x is derived. In the last section we show how the Fock bundle is constructed in the case of $3+1$ dimensional Dirac–Yang–Mills theories which will be used to unify the group action of the two groups described above on a bundle which contains the determinant bundle.

Remark A. The asymptotic model we consider here has two serious drawbacks apart from the advantage that the calculations are essentially simplified. The first is that we do not know the kernel and the image of the basic projection operators, which enter the definition of the Grassmannian, explicitly. The second one is due to the fact that we have an explicit basis for the vectors of the normal spaces to the isospectral set but that this basis is related to the $L^2_{\mathbb{R}}[0,1]$-basis by operators of the form $1+$ Hilbert–Schmidt. Since we want to work explicit we decide to work in this known basis instead taking an unknown one of the form $1+$ trace class. But this forces us to consider a more general Grassmannian in order to construct the determinant bundle. That is, it is shown that the normal spaces are the elements of the Grassmannian over $L^2_{\mathbb{R}}[0,1]$ according to the splitting into even and odd functions. The group acting on the Grassmannian will have off diagonal elements which are Hilbert–Schmidt. However, in our case we consider them as elements of the Schatten class $\mathcal{I}_4$ in order to construct the determinant bundle with the explicit known basis. The Grassmannian with the Hilbert–Schmidt off diagonal terms nevertheless is dense in the more general one. Since the determinant bundle constructed over the more general Grassmannian involves regularized determinants, the group action on the bundle will be more complicated.

1. Grassmannian

Let $L^2_{\mathbb{R}}[0,1]$ be our Hilbert space which the polarization given by the subspace E of even function and of the subspace of the odd function U. Recall that the symmetry of the function is given with respect to the point $\frac{1}{2}$. The definition of the Grassmannian $Gr(H)$ of an arbitrary Hilbert space $H = H_+ \oplus H_-$ with the indicated polarization is as follows.

Definition 1. $Gr_2(H)$ is the set of all closed subspaces $W \subset H$ such that
(a) the orthogonal projection

$$pr_+ : W \to H_+ \tag{1.1}$$

is a Fredholm operator, and

(b) the orthogonal projection

$$pr_- : W \to H_- \tag{1.2}$$

is a compact operator belonging to the Schatten class $\mathcal{I}_4$.

We only consider the connected component with index 0 of the operator pr_+. We state the properties of such Grassmannians which are all proved in the book "Loop Groups" of Pressley and Segal in [10, Ch. 7].

Theorem 2.

(a) *$Gr_2(H)$ is a Hilbert manifold modeled on $\mathcal{I}_4(W, W^\top)$, i.e. it is modeled over the space of operators from W to $W^\top$ belonging to $\mathcal{I}_4$.*

(b) *The group $GL_{\text{res}}(H)$ acts transitively on $Gr(H)$.*

The restricted general linear group $GL_2(H)$ is the following closed subgroup of the general linear group $GL(H)$ of invertible operators on H: Let $g \in GL(H)$ be written in block form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.3}$$

according to the polarization of the Hilbert space. Then $g \in GL_{\text{res}}(H) \iff b, c \in \mathcal{I}_4$. Note that in this case a, d are automatically the Fredholm operators. This follows from the fact that $W \in Gr_2(L^2_{\mathbb{R}}[0,1]) \iff W$ is equal to the image of the embedding $w_+ \oplus w_- : E \to E \oplus U$ with w_+ Fredholm and $w_- \in \mathcal{I}_4$. Then

$$g \cdot w = g \cdot (w_+ \oplus w_-) = g \cdot \begin{pmatrix} w_+ \\ w_- \end{pmatrix} = \begin{pmatrix} w'_+ \\ w'_- \end{pmatrix} = \begin{pmatrix} aw_+ + cw_- \\ bw_+ + dw_- \end{pmatrix}$$

and w'_+ is Fredholm as the sum of a Fredholm and a Hilbert–Schmidt operator and w'_- is Hilbert–Schmidt as being the sum of the two Hilbert–Schmidt operators. We prove that the normal space $N_q(M(p))$ at the isospectral set for every $q \in L^2_{\mathbb{R}}[0,1]$ is an element of $Gr_2(L^2_{\mathbb{R}}[0,1])$.

Lemma 3. *$N_q(M(p))$ is for every $q \in L^2_{\mathbb{R}}[0,1]$ an element of $Gr_{2,[q]}(L^2_{\mathbb{R}}[0,1])$, where the index $[q]$ denotes that the isospectral set $M(p)$ is a real analytic submanifold of $L^2_{\mathbb{R}}[0,1]$ lying in the hyperplane of all functions with mean $[q] := \int_0^1 q(t)\,\mathrm{d}t$.*

We omit the index $[q]$ indicating the hyperplane with the understanding that we keep it fixed for the moment.

Proof. Let $u = \sum_{n \geq 0} \eta_n U_n$, $\eta \in \mathbb{R} \times \ell^2$. We have to prove that pr_+ and pr_- are the Fredholm and Hilbert–Schmidt operators from $N_q(M(p))$ to E and U, respectively,

$$pr_+u = \sum_{n=0}^{\infty} \langle u, \cos 2\pi nx \rangle \cos 2\pi nx,$$

$$\sum_{n,m=0}^{\infty} \eta_m \langle g_m^2 - 1, \cos 2\pi nx \rangle \cos 2\pi nx \tag{1.4}$$

$$= \sum_{n,m=0}^{\infty} \eta_m (\delta_{mn} + \langle O(1/m), \cos 2\pi nx \rangle) \cos 2\pi nx,$$

where we used (1.7) from [20].

But by the Bessel inequality,

$$\sum_{n,m\geq 0}^{\infty} |\langle O(1/m), \cos 2\pi nx \rangle|^2 \leq \sum_{m\geq 0}^{\infty} |O(1/m)|^2 \leq c \sum_{m\geq 0}^{\infty} \frac{1}{m^2} \leq \infty. \tag{1.5}$$

Hence, the operator pr_+ is of the form

$$pr_+ = 1 + S, \tag{1.6}$$

with S Hilbert–Schmidt. Since the Fredholm operators are invertible modulo compact operators, we proved that pr_+ is Fredholm. We omit the same proof for pr_-, which gives us that pr_- is a Hilbert–Schmidt operator. □

We proved that in fact the Grassmannian we consider here is Gr_1, that is with the Hilbert–Schmidt operators in the off diagonal terms of the group action w.r.t. the polarization. Since the Hilbert–Schmidt operators also belong to the Schatten class $\mathcal{I}_4$, there is a dense embedding of Gr_1 in Gr_2. Since in the construction of the determinant bundle we will need what is called an admissible basis and in our case the admissible basis, we explicitly now in order to do calculations leading to Gr_2 we will consider this Grassmannian (see Remark A). In Theorem 2 we stated that the Grassmannian is a Hilbert manifold. We give now explicit coordinate charts. To get them we imitate the procedure for $L^2(S^1, \mathbb{C})$ in [10, p. 103]. Let $\{e_k\}_{k\in\mathbb{Z}, k>0}$ be the basis vectors for the even subspace E of $L^2_{\mathbb{R}}[0, 1]$ and $\{e_{-k}\}_{k\in\mathbb{Z}, k<0}$ those for the odd subspace U. Then S is a subset of $\mathbb{Z}$ which is bounded below and if the number of negative basis vectors is $m \in \mathbb{N}$, then all positive numbers but the first m ones belong to m. We make a finite dimensional example which shows the coherence of the above definition of the infinite dimensional Grassmannian with the finite dimensional one. Let $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$ be the Hilbert space corresponding to $L^2_{\mathbb{R}}[0, 1]$. Then the Grassmannian Gr_4 is the manifold whose elements are all subspaces of $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$. This means that $Gr_4 = \cup_{i=1}^4 Gr_{4,i}$, i.e. it is the disconnected sum of the one, two, three and four dimensional subspaces. The number of charts which are necessary to parametrize this subspaces is equal to

$$\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 4 + 6 + 4 + 1 = 15.$$

This number can be put into relation with the set $\mathcal{S}$. If we denote the basis vectors of $\mathbb{C}^4$ by $e_{-1} =: -1, e_{-2} =: -2, e_1 =: 1, e_2 =: 2$, where those with the same signs span one of the subspaces $\mathbb{C}^2$. Then the set $\mathcal{S}$ consists of the spaces spanned by the following subspaces of $\mathbb{C}^4$:

$S_1 = \{-1, -2, 1, 2\}$, corresponding to $\mathbb{C}^4$;

$S_2, \ldots, S_5 = \{-1\}, \{-2\}, \{1\}, \{2\}$,
corresponding to the one dimensional subspaces;

$S_6, \ldots, S_{11} = \{-2, -1\}, \{-2, 1\}, \{-2, 2\}, \{-1, -1\}, \{-1, 2\}, \{1, 2\}$,
corresponding to the two dimensional subspaces;

$S_{12}, \ldots, S_{15} = \{-2, -1, 1\}, \{-2, -1, 2\}, \{-1, 1, 2\}, \{-2, 1, 2\}$,
corresponding to the three dimensional subspaces.

It is clear that to any projection of $W \in Gr_4$ to $\mathbb{C}^4$ there is a subset $S \in \mathcal{S}$ such that this projection is an isomorphism.

Let $\mathcal{S}$ be the collection of all this sets S and H_S is the closed subspace spanned by S. Using this fact the following lemma is easy to prove [10, p.103].

Lemma 4. *For any $W \in Gr_2(L^2_{\mathbb{R}}[0, 1])$ there is a set $S \in \mathcal{S}$ such that the orthogonal projection $W \to H_S$ is an isomorphism. In other words the sets $\{U_{H_S}\}_{S \in \mathcal{S}}$, where*

$$U_{H_S} = \begin{array}{l}\{\hat{W} \in Gr(L^2_{\mathbb{R}}[0, 1]) \mid p : \hat{W} \to H_S,\\ p \text{ an orthogonal projection and isomorphism}\},\end{array} \tag{1.7}$$

form an open covering of $Gr_2(L^2_{\mathbb{R}}[0, 1])$.

2. Group acting on the Grassmannian for fixed parameter x

In the first part, we describe the general situation, i.e. the non-asymptotic case, where we keep $x \in [0, 1]$ in the normal vectors $U(x,)$ fixed. Using that $E \in Gr_2(L^2_{\mathbb{R}}[0, 1])$ and that we only consider the connected component with index$(pr_+) = 0$, there exists a group G_2, which according to the splitting of the Hilbert space any element has the form

$$g = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad g \in G_2,$$

where any element maps E into a normal space at a point $q \in M(p)$ and a, d are the Fredholm operators, $b, c \in \mathcal{I}_4$, respectively. That is, there exists a $q \in G_2$ such that

$$g \cdot E = N_q(M(p)). \tag{2.1}$$

Clearly, g is a function of x and q. Fig. 1 illustrates the geometric setting. How this group acts explicitly on the Grassmannian? Clearly, the following holds. Let $v \in E$ be a vector in the normal space at a point $q \in E$. With respect to the polarization $L^2_{\mathbb{R}}[0, 1] = E \oplus U$ v has a zero component w.r.t. U. Let then $g \in G_2$ acting on v, i.e.

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \begin{pmatrix} pr_+u \\ pr_-u \end{pmatrix} \in N_q(M(e)). \tag{2.2}$$

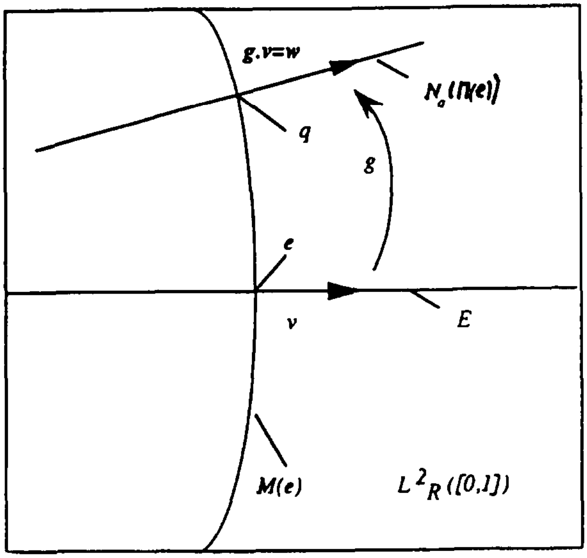

Fig. 1.

Inserting the explicit formula for $u \in N_q(M(e))$ written down in (1.7) and using the asymptotic expansion (1.6),

$$(2.2) = \begin{pmatrix} \sum_{n\geq 0}^{\infty} \eta_n pr_+ U_n \\ \sum_{n\geq 0}^{\infty} \eta_n pr_- U_n \end{pmatrix} = \begin{pmatrix} \sum_{n\geq 1}^{\infty} \eta_n pr_+ (g_n^1 - 1) + \eta_0 pr_+ 1 \\ \sum_{n\geq 1}^{\infty} \eta_n pr_- (g_n^2 - 1) + \eta_0 pr_+ 1 \end{pmatrix}. \tag{2.3}$$

Since $v \in E$,

$$v = \sum_{n=0}^{\infty} \langle v, \cos 2\pi nx \rangle \cos 2\pi nx, \tag{2.4}$$

and we get for the nth basis vector of v the equation ($m > 1$)

$$\begin{aligned} \begin{pmatrix} \sum_n a_{mn} \cos 2\pi nx \\ \sum_n b_{mn} \cos 2\pi nx \end{pmatrix} &= \begin{pmatrix} \eta_m pr_+ (g_m^2 - 1) \\ \eta_m pr_- (g_m^2 - 1) \end{pmatrix} \\ &= \begin{pmatrix} \eta_m pr_+ (\mathrm{O}(1/m) - \cos 2\pi mx) \\ \eta_m pr_- (\mathrm{O}(1/m) - \cos 2\pi nx) \end{pmatrix} \\ &= \begin{pmatrix} \eta_m pr_+ (\mathrm{O}(1/m) - \cos 2\pi mx) \\ \eta_m pr_- \mathrm{O}(1/m) \end{pmatrix}. \end{aligned} \tag{2.5}$$

To find a solution of (2.5), i.e. to determine a and b in a direct way seems hopeless. We try an indirect way and instead of finding the explicit form of the group action on an element of E we consider the general case with $v \in N_p(M(e))$, $p \neq e, e \in E$. That is v has also two components v_+ and v_-. We first consider the operators a, b, c, d of $g \in G_2$ in

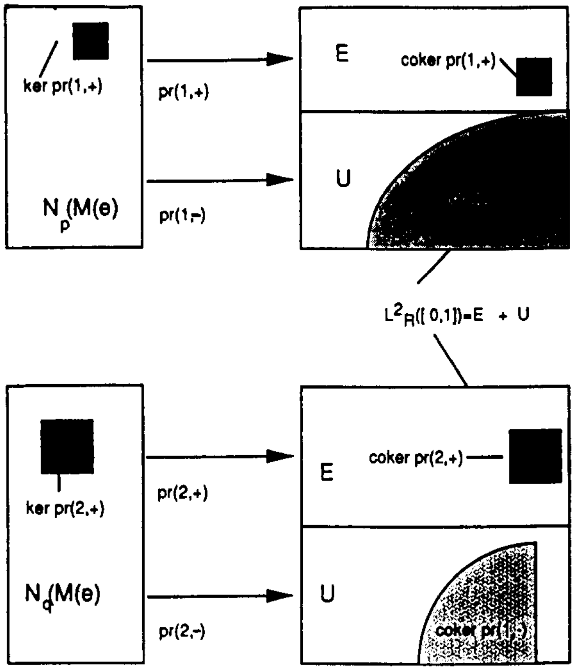

Fig. 2.

more detail. Let $w \in N_q(M(e))$ such that w is reached from v by g. Furthermore, the two components of w according to the polarization are w_+ and w_-. Let $\epsilon_{2m} = \cos 2\pi mx$ and $\omega_{2m} = \sin 2\pi x$ denote the basis vectors of E and U, respectively. Fig. 2 shows the projection of the two elements $N_p(M(e))$ and $N_q(M(e))$ of the Grassmannian into the space $E \oplus U$. We consider this image since we study the group element g transporting v into w as a map of the appropriate subspaces of $L^2_{\mathbb{R}}[0, 1]$ into itself. Fig. 2 shows the various spaces and maps of the following discussion.

Since $pr_+^{1,2}$ are Fredholm, we have that

$$\dim \ker p_+^1 = \dim \operatorname{coker} p_+^1, \qquad \dim \ker p_+^2 = \dim \operatorname{coker} p_+^2. \tag{2.6}$$

Then the group element g is a linear bijective map from

$$g : E \oplus U \ominus \operatorname{coker} pr_+^1 \ominus \operatorname{coker} pr_-^1 \to E \oplus U \ominus \ker pr_+^2 \ominus \ker pr_-^2. \tag{2.7}$$

Furthermore,

$$a : E \to E, \qquad d : U \to U, \qquad b : E \to U, \qquad c : U \to E. \tag{2.8}$$

Since $a \in \mathcal{F}$, where $\mathcal{F}$ denotes the class of Fredholm operators, we may write it in the form

$$a = q + t \quad \text{with } q \text{ invertible and } t \text{ finite rank.} \tag{2.9}$$

q is an invertible map from $pr_+^1(N_p(M(e)))$ in $pr_+^2(N_q(M(e)))$. t maps a finite dimensional subspace $M \subset E$ into a finite dimensional subspace $t(M) =: L$ of $pr_+^2(N_p(M(e)))$ with

$\dim L \le M$. The coker K of t has dimension $\dim K = \dim M - \dim L$. The operator c maps $pr_-^1(N_p(M(e)))$ in $pr_+^2(N_p(M(e)))$. Since g is invertible, $a+c$ has to be surjective on $pr_+^2(N_q)$, hence c maps a subspace $J \subset pr_-^1(N_p)$ in $pr_+^2(N_q)$, with $\dim J = \dim K$. For b, c we have: $d \in \mathcal{F}$ maps all but a finite subspace G of $pr_-^1(N_p)$ bijective in $pr_-^2(N_q)$ and b is Hilbert–Schmidt which maps $pr_+^1(N_p)$ in $pr_-^2(N_q)$ where there is a finite dimensional subspace H of $pr_+^1(N_p)$, $\dim H = \dim G - \dim \operatorname{coker} g$, which is mapped in G since, again, g has to be invertible.

The concrete realization of g is now given in the asymptotic case.

Theorem 5. *Let*

$$U(x, p(x)) \in N_p(M(e)), \qquad h := h(x, p(x)) = 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log\det \Theta(x, tq, \xi) \in M(e)$$

with

$$|\langle f, f\rangle|, \qquad |\langle h, f\rangle| \le \|h\|_2, \quad 1 < \|h\|_2, \qquad \forall f \in L^2_{\mathbb{R}}[0,1].$$

Then the elements of the group G_2 which acts on the Grassmannian $Gr_2(L^2_{\mathbb{R}}[0,1])$ are given by $g(x, tq) = (1, \mathcal{B}(x, tq(x)))$, where $\mathcal{B}(x, tq(x))$ is the operator defined by the equation

$$\mathcal{B}(x, tq(x))U(x, p(x)) = 2\sum_{n\ge 1} \eta_n \mathrm{O}\Big(\frac{1}{n}\Big) \sum_k \frac{1}{k!},$$

$$\sum_{m=0}^{k} \binom{k}{m} (\sqrt{x})^m P_m(h, h, \ldots, h; p) P_{m-k}(h, h, \ldots, h; p) =: (\mathrm{e}^{-\partial_h} - 1)U(x, p(x)),$$

and the sum converges if the above bounds on h are imposed. The functions P_k are defined by

$$\frac{\partial^k}{\partial h^k(t)} g_n(x, h, t)(f_1, f_2, \ldots, f_k)(f) =: \partial_h^k g_n = (\sqrt{x})^k P_k(f_1, f_2, \ldots, f_k; f)\mathrm{O}\Big(\frac{1}{n}\Big),$$

and their specific form is given in the proof.

Remark. The identity element of the group G_2 is $e = (1, 0)$, the inverse element g^{-1} to the element $g(x, tq) = (1, \mathcal{B}(x, tq(x)))$ is given by $g^{-1}(x, tq) = (1, -\mathcal{B}(x, tq(x)))$ and the composition of two elements g_1 and g_2 is defined by

$$g_1(x, sw(x)) * g_2(x, tq(x)) = (1, \mathcal{B}(x, sw(x))\mathcal{B}(x, tq(x))).$$

Remark. After the proof of Theorem 5 we give as a corollary the explicit form of the group elements when we consider the polarization of the Hilbert space.

Proof. The strategy will be a perturbation of the normal vector at $q \in M(e)$ in a direction $h \in M(e)$. We start with some facts which are taken from [11]. On the isospectral set $M(e)$ there exists an addition of two points, i.e.

$$w \oplus p =: \exp_e(V_{\kappa(w)+\kappa(p)}) \tag{2.10}$$

for $w, p \in M(e)$. The even point $e \in M(e)$ is the identity element of the group with the group operation $\oplus$ and the inverse element of p is given by reflection at the space subspace of even functions E. Furthermore, a curve on the isospectral set is denoted by $\Phi^t(q, V_\xi)$, where $\Phi^0 = q, a < t < b$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t(q, V_\xi) = V_\xi(\Phi^t(q, V_\xi)), \qquad \exp_q(V_\xi) = \Phi^t(q, V_\xi)|_{t=1}. \tag{2.11}$$

A point on the isospectral set for fixed t is given by

$$\Phi^t(q, V_\xi) = e - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\det\Theta(x, tq, \xi), \tag{2.12}$$

where

$$\Theta(x, tq, \xi) = \left(1 + (\mathrm{e}^{t\xi_i} - 1)\int_x^1 g_i(s, q)g_j(s, q)\,\mathrm{d}s\right)_{i,j\in\mathbb{N}}. \tag{2.13}$$

Since the derivative of a determinant is given by the formula

$$\frac{\mathrm{d}}{\mathrm{d}s}\det A(s) = \det A(s)\mathrm{tr}\left(\frac{\mathrm{d}}{\mathrm{d}s}A(s)\right)A^{-1}(s),$$

we get

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\det\Theta(x, tq, \xi) = \mathrm{tr}\Bigg(&\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Theta(x, tq, \xi)\right](\Theta(x, tq, \xi))^{-1}\\ &+\left[\frac{\mathrm{d}}{\mathrm{d}x}\Theta(x, tq, \xi)\right](\Theta(x, tq, \xi))^{-2}\Bigg).\end{aligned} \tag{2.14}$$

Is the determinant well defined? The asymptotic expansion of g_j is

$$g_j(x, \lambda_j, q) = \sqrt{2}\sin\pi jx + \mathrm{O}(1/j). \tag{2.15}$$

Hence,

$$\begin{aligned}&\int_x^1 g_i(s, q)g_j(s, q)\,\mathrm{d}s\\ &= -\frac{2}{\pi(i^2 - j^2)}(1 - \delta_{ij})[j\sin\pi ix\cos\pi jx - i\sin\pi jx\cos\pi ix]\\ &+[1 - x + \frac{1}{4\pi i}\sin 2\pi ix]\delta_{ij} + \mathrm{O}\left(\frac{1}{ij}\right) + \mathrm{O}\left(\frac{1}{j}\right) + \mathrm{O}\left(\frac{1}{i}\right)\\ &=: \delta_{ij} + F(x)_{ij}.\end{aligned} \tag{2.16}$$

Therefore,

$$\int_x^1 g_i(s,q)g_j(s,q)\,\mathrm{d}s = \delta_{ij} + F(x)_{ij} + \mathrm{O}(1/ij), \tag{2.17}$$

and

$$\Theta := \lim_{n\to\infty} \det \Theta^{(n)}$$

$$\det \Theta^{(n)} = \left\{\prod_i^{\infty} \mathrm{e}^{\xi_i}\right\} \det[\delta_{ij} + (1-\mathrm{e}^{-t\xi_i})F(x)_{ij}]_{1\le i,\, j\le n}$$

$$=: \left\{\prod_i^{\infty} \mathrm{e}^{t\xi_i}\right\} \det GF(x)_{ij})_{1\le i,\, j\le n} =: \det \Xi \det GF. \tag{2.18}$$

The infinite determinants in (2.18) exist since

$$(t\xi_i)_{i\in\mathbb{N}} \in \ell_2 \times \mathbb{R}, \tag{2.19}$$

and

$$\{(\delta_{ij} - \mathrm{e}^{-\xi_i})F(x)_{ij}\}_{1\le i,\, j\le n}$$

$$= \underbrace{\{\sqrt{|xi_i|}\delta_{ij}\}_{1\le i,\, j\le n}}_{A} \underbrace{\left\{\frac{(1-\mathrm{e}^{-\xi_i})F(x)_{ij}}{\sqrt{|xi_i|}}\right\}_{1\le i,\, j\le n}}_{B} = AB. \tag{2.20}$$

But $A \in \mathcal{I}_2$ and since

$$\lim_{s\to 0} \frac{1-\mathrm{e}^{-s}}{\sqrt{|s|}} = 0,$$

we get with the Bessel inequality and the Parseval identity that B is also Hilbert–Schmidt. Hence, we have a product of two Hilbert–Schmidt operators which is a trace class operator. That is the determinants in (2.20) are the well defined Fredholm determinants in the limit $n \to \infty$. We further need the norm of the points on the curves on the isospectral set, i.e. [11]

$$\|\Phi^t(q,V_\xi)\| = \|tq\| = \|e\| + 8\sum_{n\ge 1} \delta_n(tq)[\cosh(\kappa_n(q) + t\xi_n) - \cosh(\kappa_n(q))], \tag{2.21}$$

where

$$\delta_n(q) = 2n^2\pi^2(1 + \mathrm{O}(\log n/n)).$$

The group G_2 for a *fixed* x acts on a normal vector $U(x,p(x))$ in the normal space $N_p(M(e))$ and transports it in the normal vector $U(x,tq)$, i.e.

$$U(x,tq) = g(x,tq)U(x,p(x)), \quad g \in G_2. \tag{2.22}$$

Remark. As a shorthand we use the notation $U(x,tq) = U(x, \Phi^t(q))$.

Using Eq. (2.12), we write

$$U(x,tq) = U(x, p(x) - h(x, p(x))), \qquad h(x, p(x)) = 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log\det \Theta(x, tq, \xi). \tag{2.23}$$

We formally expand $U(x, p(x) - h(x, p(x)))$,

$$\begin{aligned} &U\,(x, (p(x) - h(x, p(x), t))) \\ &= U(x, p(x)) - \left\langle \frac{\partial U(x, h(x, p(x), t))}{\partial h(x, p(x), t)}\bigg|_{h(x,\,p(x),\,t)=p(x)}, h(x, p(x), t) \right\rangle \\ &\quad + \frac{1}{2}\left\langle \frac{\partial^2 U(x, h(x, p(x), t))}{\partial h^2(x, p(x), t)}\bigg|_{h(x,\,p(x),\,t)=p(x)}, h^2(x, p(x), t) \right\rangle + \cdots - \cdots \\ &=: \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \left\langle \frac{\partial^n U(x, h(x, p(x), t))}{\partial h^n(x, p(x), t)}\bigg|_{h(x,\,p(x),\,t)=p(x),}\ h^n \right\rangle, \end{aligned} \tag{2.24}$$

and where

$$\begin{aligned} &\left\langle \frac{\partial U(x, h(x, p(x), t))}{\partial h(x, p(x), t)}\bigg|_{h(x,\,p(x),\,t)=p(x)}, h(x, p(x), t) \right\rangle \\ &= \frac{\partial U(x, h(x, p(x), t))}{\partial h(x, p(x), t)}\bigg|_{h(x,\,p(x),\,t)=p(x)} h(x, p(x), x). \end{aligned} \tag{2.25}$$

The derivative w.r.t. h is defined in the following way: First let f be a functional between a Hilbert space E and the complex numbers. If we denote the derivative of f at the point x by $\mathrm{d}_x f$, then by the Riesz representation theorem there exists a unique element $\partial f/\partial x$ in the Hilbert space E, such that $\mathrm{d}_x f(v) = \langle \partial f/\partial x, v\rangle$ for all $v \in E$. But in our case $\langle \partial U/\partial h, h\rangle$ is the tangent map from the tangent space at $q \in L^2_{\mathbb{R}}[0, 1]$, which clearly is isomorphic to $L^2_{\mathbb{R}}[0, 1]$, into $L^2_{\mathbb{R}}[0, 1]$ evaluated at h. The kth-derivative term, $\langle \partial^k U/\partial h^k, h^k\rangle$ is the k-linear map from $L^2_{\mathbb{R}}[0, 1] \times L^2_{\mathbb{R}}[0, 1] \times \cdots \times L^2_{\mathbb{R}}[0, 1]$ into $L^2_{\mathbb{R}}[0, 1]$ evaluated at $h \otimes h \otimes \cdots \otimes h$ which simply gives $(\partial^k U/\partial h^k) h^k$. We write the expansion of $U(x, (p(x) - h(x, p(x), t)))$, i.e. (2.24), in the form

$$U(x, p(x) - h(x, p(x), t)) := \left\langle \exp\left[\frac{\partial}{\partial h}\right] U(x, (p(x)), \exp[h]) \right\rangle. \tag{2.26}$$

We need to calculate the derivative of $U(x, h(x, p(x), t))$ w.r.t. $h(x, p(x), t)$.

Lemma 6.

(a) $$\frac{\partial}{\partial h(x, p(x), t)} g_n(x, h(x, p(x), t))(f) =: \partial_h g_n$$

$$= \sqrt{x}\frac{\langle h, f\rangle}{\|h\|} \mathrm{O}\left(\frac{1}{n}\right) = \mathrm{O}\left(\frac{1}{n}\right). \tag{2.27}$$

(b)
$$\frac{\partial^k}{\partial h^k(t)} g_n(x,h,t)(f_1, f_2, \ldots, f_k)(f) =: \partial_h^k g_n$$
$$= (\sqrt{x})^k P_k(f_1, f_2, \ldots, f_k; f)\mathrm{O}(1/n) = \mathrm{O}(1/n), \tag{2.28}$$

for $k > 1$ and where the notation $(\partial/\partial h) g_n(x,h)(g)$ is defined by

$$\frac{\partial}{\partial h} g_n(x,h)(f) := \frac{\mathrm{d}}{\mathrm{d}t} g_n(x, h+tf)\Big|_{t=0},$$

and $\langle\ ,\ \rangle$ denotes the L^2-inner product.

In (2.28) the first three functions $P_k(f_1, f_2, \ldots, f_k; h)$ have the following expressions:

$$P_1(f;h) = \frac{\langle f,h\rangle}{\|h\|},$$
$$P_2(f_1,f_2;h) = \frac{\langle f_1,f_2\rangle}{\|h\|} - \frac{\langle h,f_1\rangle\langle h,f_2\rangle}{\|h\|^3},$$
$$P_3(f_1,f_2,f_3;h) = \frac{-\langle f_1,f_2\rangle\langle f_3,h\rangle - \langle f_3,f_2\rangle\langle f_1,h\rangle - \langle f_1,f_3\rangle\langle f_2,h\rangle}{\|h\|^3} + 3\frac{\langle h,f_1\rangle\langle h,f_2\rangle\langle h,f_3\rangle}{\|h\|^5}.$$

The proof of this lemma is given in Appendix A.

We continue with the proof of Theorem 5. Lemma 6 and formulae (2.24), (2.27) and (2.28) imply for the derivative of the normal vector $U(x, h(x, p(x), t))$

$$\left\langle \frac{\partial U(x,h(x,p(x),t))}{\partial h(x,p(x),t)}\Big|_{h=p}, h\right\rangle = 2\sum_{n\geq 1} \eta_n \sqrt{x}\frac{\langle p,h\rangle}{\|p\|}\mathrm{O}\Big(\frac{1}{n}\Big), \tag{2.29}$$

and for the higher derivatives,

$$\left\langle \frac{\partial^k U(x,h(x,p(x),t))}{\partial h^k(x,p(x),t)}\Big|_{h=p}, h^k\right\rangle = 2\sum_{n\geq 1} \eta_n \sum_{m=0}^{k} \binom{k}{m} (\sqrt{x})^m P_m(h,h,\ldots,h;p)$$
$$\times P_{k-m}(h,h,\ldots,h;p)\mathrm{O}\Big(\frac{1}{n^2}\Big). \tag{2.30}$$

We next give a graphical description of the terms $P_m(h,h,\ldots,h;f)$. The first terms are

$$P_1 = \frac{\langle f,h\rangle}{\|h\|}, \qquad P_2 = \frac{\langle f,f\rangle}{\|h\|} - \frac{\langle f,h\rangle^2}{\|h\|^3},$$
$$P_3 = -3\frac{\langle f,f\rangle\langle f,h\rangle}{\|h\|^3} + 3\frac{\langle f,h\rangle^3}{\|h\|^5},$$
$$P_4 = -3\frac{\langle f,f\rangle^2}{\|h\|^3} + 18\frac{\langle f,f\rangle\langle f,h\rangle^2}{\|h\|^5} - 15\frac{\langle f,h\rangle^4}{\|h\|^7},$$
$$P_5 = 45\frac{\langle f,f\rangle^2\langle f,h\rangle}{\|h\|^5} - 150\frac{\langle f,f\rangle\langle f,h\rangle^3}{\|h\|^7} + 105\frac{\langle f,h\rangle^5}{\|h\|^9},$$

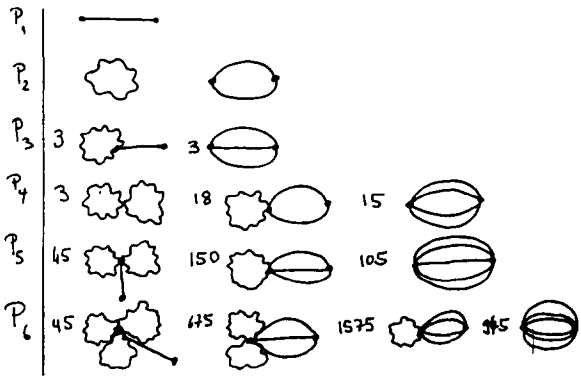

Fig. 3. Graphical representation of P_1 to P_6

$$P_6 = 45\frac{\langle f,f\rangle^3}{\|h\|^5} - 675\frac{\langle f,f\rangle^2\langle f,h\rangle^2}{\|h\|^7} + 1575\frac{\langle f,h\rangle^4\langle f,f\rangle}{\|h\|^9} - 945\frac{\langle h,f\rangle^6}{\|h\|^{11}}.$$

If we write a bubble for $\langle f,f\rangle$ and the line for $\langle h,f\rangle$, we get the following graphical representations for P_1 to P_6 with the multiplicities modulo the $\|h\|^{-k}$-factors (Fig. 3).

If we use Θ for $\langle h,f\rangle$ and Ω for $\langle f,f\rangle$, respectively, we can write P_{2n+1} in the form

$$P_{2n+1} = \sum_{i+2j=2n+1,\, q\in\mathcal{P}_o} b_q^{(2n+1)}\Theta^i\Omega^{2j},$$

where $\mathcal{P}_0$ is the finite subset of the natural numbers consisting of all solutions of the equation $i+2j=2n+1$ w.r.t. $i\in\mathbb{N}$ and $j\in\mathbb{N}$ for fixed n.

Example.

$$P_5 = b_1^{(5)} + b_2^{(5)} + b_3^{(5)},$$

and comparing this with the above table we get

$$b_1^{(5)} = \frac{45}{\|h\|^5}, \qquad b_2^{(5)} = \frac{105}{\|h\|^9}, \qquad b_3^{(5)} = -\frac{150}{\|h\|^7}.$$

In the same way we have the decomposition

$$P_{2n} = \sum_{i+2j=2n+1,\, q\in\mathcal{P}_e} b_q^{(2n)}\Theta^i\Omega^{2j},$$

where $\mathcal{P}_e$ is the finite subset of the natural numbers consisting of all solutions of the equation $i+2j=2n$ w.r.t. $i\in\mathbb{N}$ and $j\in\mathbb{N}$ for fixed n. We now give necessary conditions $\langle f,f\rangle, \langle h,f\rangle$ and $\|h\|$ have to satisfy such that

$$\sum_{k>0}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(\sqrt{x})^m P_m(h,h,\ldots,h;f)P_{k-m}(h,h,\ldots,h;f)$$

converges.

Let condition (∗) be:

$$(*) \qquad |\langle f, f\rangle|, \qquad |\langle h, f\rangle| \leq \|h\|, \quad 1 < \|h\|.$$

This implies that

$$|P_n(h, h, \ldots, h; f)| \leq \frac{1}{\|h\|^{2n-1}} \sum_{k=0}^{n} s_k \frac{1}{\|h\|^k},$$

and if we write s_k^* for maximal absolute value of the coefficients in the graphs belonging to P_k, we have $|P_n(h, h, \ldots, h; f)| \leq cs_k^*/\|h\|^{2n-1}$, where c is a constant. Hence,

$$\left|\sum_{k>0} \frac{1}{k!} \sum_{m=0}^{k} \binom{k}{m} (\sqrt{x})^m P_m(h, h, \ldots, h; f) P_{k-m}(h, h, \ldots, h; f)\right|$$
$$\leq c \sum_{k>0} \frac{1}{\|h\|^{2k-1}} \left(\sum_{m=0}^{k} \left(\frac{1}{(k-m)!}\right)^2\right)^2 \left(\sum_{m=0}^{k} \left(\frac{|s_m^*|}{m!}\right)^2\right)^2,$$

where we applied the Hölders inequality in the last line. We further get,

$$c \sum_{k>0} \frac{1}{\|h\|^{2k-1}} \left(\sum_{m=0}^{k} \left(\frac{1}{(k-m)!}\right)^2\right)^2 \left(\sum_{m=0}^{k} \left(\frac{|s_m^*|}{m!}\right)^2\right)^2$$
$$\leq c \sum_{k>0} \frac{1}{\|h\|^{2k-1}} \left(\sum_{m=0}^{k} \left(\frac{|s_m^*|}{m!}\right)^2\right)^2.$$

A sufficient condition for the last integral is given by the inequality

$$\frac{1}{\|h\|^{2k-1}} \left(\sum_{m=0}^{k} \left(\frac{|s_m^*|}{m!}\right)^2\right)^2 < \frac{1}{k^2},$$

or equivalently,

$$\|h\| > \left(k \sum_{m=0}^{k} \left(\frac{|s_m^*|}{m!}\right)^2\right)^{\frac{2}{2k-1}}.$$

Since $(n)^{1/n} \to 1$, for $n \to \infty$, the sufficient condition

$$\left(\sum_{m=0}^{k} \left(\frac{|s_m^*|}{m!}\right)^2\right)^{\frac{2}{2k-1}} \to 1$$

is fulfilled, if $\sum_{m=0}^{k}(|s_m^*|/m!)^2 \sim k^\alpha, \alpha \in \mathbb{R}$. But the leading order coefficient $s_k^*, \forall k$, is smaller than k^k, which implies that

$$\sum_{m=0}^{k} \left(\frac{|s_m^*|}{m!}\right)^2 \leq \sum_{m=0}^{k} \left(\frac{m^m}{m!}\right)^2 \leq c \sum_{m=0}^{k} m.$$

This shows that $\sum_{m=0}^{k}(|s_m^*|/m!)^2 \sim k^{\alpha}$, and therefore conditon $(*)$ is sufficient for

$$\left|\sum_{k>0}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(\sqrt{x})^m P_m(h,h,\ldots,h;f)P_{k-m}(h,h,\ldots,h;f)\right|$$

being finite. In the same way one proves that for $\|h\| < 1$ and $\langle h,f\rangle^{k+2}, \langle f,f\rangle^{k+2} < \|h\|$, $k > 2$, the sum over the graphs converges.

We now return to the group action!

Lemma 6 and formulae (2.24), (2.27), (2.28) and (2.30) give as a final result for the shifted vector $U(x,(p(x)-h(x,p(x),t)))$ the formula

$$\begin{aligned} &U(x,(p(x)-h(x,p(x),t)))\\ &\quad= U(x,p(x)) + 2\sum_{n\geq 1}\eta_n \mathrm{O}\Big(\frac{1}{n}\Big)\sum_{k}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}\\ &\qquad\times(\sqrt{x})^m P_m(h,h,\ldots,h;p)P_{m-k}(h,h,\ldots,h;p). \end{aligned} \tag{2.31}$$

Writing a group element in the form $g(x,tq) = (1,\mathcal{B}(x,tq(x)))$, where $\mathcal{B}(x,tq(x))$ is the operator defined by the equation

$$\begin{aligned} &\mathcal{B}(x,tq(x))U(x,p(x))\\ &\quad= 2\sum_{n\geq 1}\eta_n \mathrm{O}\Big(\frac{1}{n}\Big)\sum_{k}\frac{1}{k!}\sum_{m=0}^{k}\binom{k}{m}(\sqrt{x})^m P_m(h,h,\ldots,h;p)P_{m-k}(h,h,\ldots,h;p)\\ &\quad= (\mathrm{e}^{-\partial_h}-1)U(x,p(x)). \end{aligned} \tag{2.32}$$

The identity element of the group G_2 is $e = (1,0)$, the inverse element g^{-1} to the element $g(x,tq) = (1,\mathcal{B}(x,tq(x)))$ is given by $g^{-1}(x,tq) = (1,-\mathcal{B}(x,tq(x)))$ and the composition of two elements g_1 and g_2 is defined by

$$g_1(x,sw(x)) * g_2(x,tq(x)) = (1,\mathcal{B}(x,sw(x))\mathcal{B}(x,tq(x))). \tag{2.33}$$

This proves Theorem 5. □

We now discuss the form of the group action w.r.t. the polarization of the Hilbert space for fixed x. That is we have to determine the matrix elements a, b, c and d in the equation

$$\begin{pmatrix} U_+(x,tq(x)) \\ U_-(x,tq(x)) \end{pmatrix} = \begin{pmatrix} a(x,tq(x)) & c(x,tq(x)) \\ b(x,tq(x)) & d(x,tq(x)) \end{pmatrix}\begin{pmatrix} U_+(x,p(x)) \\ U_-(x,p(x)) \end{pmatrix}. \tag{2.34}$$

(2.34) implies that

$$\begin{aligned} aU_+(x,p(x)) + cU_-(x,p(x)) &\in \ker pr_-,\\ cU_+(x,p(x)) + dU_-(x,p(x)) &\in \ker pr_+. \end{aligned} \tag{2.35}$$

At this point the use of the asymptotic analysis breaks down, since the determination of the kernel of the orthogonal projections from the normal spaces to the even and odd subspaces of $L^2_{\mathbb{R}}[0,1]$ need a non-asymptotic analysis. A special solution which satisfies the condition (2.35) is given by

$$g(x,tq) = \begin{pmatrix} 1 + pr_+\mathcal{B}(x,tq(x)) & pr_+\mathcal{B}(x,tq(x)) \\ pr_-\mathcal{B}(x,tq(x)) & 1 + pr_-\mathcal{B}(x,tq(x)) \end{pmatrix}. \tag{2.36}$$

Note that $\mathcal{B}(x,tq(x))$ is Hilbert–Schmidt, hence $g(x,tq)$ is of the form

$$g(x,tq) = \begin{pmatrix} \mathcal{F} & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{F} \end{pmatrix}, \tag{2.37}$$

that is $g(x,tq) \in G_2$. Therefore, Theorem 5 can be written down without difficulty when we regard the group action w.r.t the polarization of the Hilbert space.

3. Determinant bundle and the tau function for fixed parameter x

For a more exhaustive discussion of the determinant bundle see [4,10]. First we introduce the groups G^2 and F_2 defined by

$$G^2 := \{A \in GL(L^2_{\mathbb{R}}[0,1]) \mid A - 1 \in \mathcal{I}_2\} \tag{3.1}$$

and

$$F_2 := \{(g,q) \mid g \in G_2,\ q \in GL(E), aq^{-1} - 1 \in \mathcal{I}_2\}, \tag{3.2}$$

where $g = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in G_2$. F_2 is a subgroup of $G_2 \times GL(E)$. The group G^2 acts on F_2 from the left in the form

$$(g,q)s := (g,qs), \quad s \in G_2, (g,q) \in F_2. \tag{3.3}$$

Let $U = \{U_n = g_n^2 - 1, n > 1,\}_{n\in\mathbb{N}}$, $U_0 = 1$ be an orthonormal basis of $N_q(M(e)) \in Gr_2(L^2_{\mathbb{R}}[0,1])$. In order to define the determinant bundle, we have to select all those bases of an element N in the Grassmannian which are related to the orthonormal basis of $L^2_{\mathbb{R}}[0,1]$ by a matrix which has a determinant. Such bases are called the admissible ones. This is the difference in the construction of the determinant bundle in the infinite dimensional case compared to the finite dimensional one (see Remark A).

Definition 7. Let $W = \{W_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $N \in Gr_2(L^2_{\mathbb{R}}[0,1])$. $W = \{W_n\}_{n\in\mathbb{N}}$ is furthermore an admissible basis of N if (a) W can be reached from the orthonormal basis of $L^2_{\mathbb{R}}[0,1]$ by a linear isomorphism and (b) if the matrix w_+ defined by

$$pr_+ w_k = \sum_{j\geq 0} (w_+)_{jk} e_j \tag{3.4}$$

is in $1 + \mathcal{I}_2$, where the set $\{e_j\}$ is an orthonormal basis of $L^2_{\mathbb{R}}[0,1]$.

The basis $U = \{U_n = g_n^2(x,q(x)) - 1, n > 1,\}_{n\in\mathbb{N}}$ is an admissible basis of $N_q(M(e))$ and the matrix w_+ is $1 + S$ in this case (see Lemma 3). The set of all admissible bases for all $N \in Gr_2(L^2_{\mathbb{R}}[0,1])$ is denoted $St_2(Gr_2(L^2_{\mathbb{R}}[0,1]))$ and is itself a manifold, called the Stiefel manifold. In order to continue we first give the definition of the regularized determinants and state some of its properties (for the proofs see [14,17]). If A is trace class, i.e. $A \in \mathcal{I}_1$, then

$$\log\det A = \operatorname{tr}\log A = \operatorname{tr}(A - \tfrac{1}{2}A^2 + \tfrac{1}{2}A^3 - \cdots). \tag{3.5}$$

Hence, if A is Hilbert–Schmidt in order that an expansion of the form (2.12) is possible, we have to subtract $\operatorname{tr} A$ which is not defined for $A \in \mathcal{I}_2$. Since the next terms in (2.12) consist of powers of A and any power A^n, $n > 1$, of a Hilbert–Schmidt operator is trace class, the definition

$$\det{}_2(1 + A) = \det((1 + A)\mathrm{e}^{-A}), \quad A \in \mathcal{I}_2, \tag{3.6}$$

gives us a well defined object which shares most of the properties a usual determinant does.

Theorem 8. *Let $A \in \mathcal{I}_2$. Then:*

(a) *The mapping $A \to \det_2(1 + A)$ is continuous in the topology of $\mathcal{I}_2$.*
(b) *$1 + A$ is invertible $\Longleftrightarrow$ $\det_2(1 + A) \neq 0$.*
(c) *If $A \in \mathcal{I}_1$, then*

$$\det{}_2(1 + A) = \det(1 + A)\mathrm{e}^{-\operatorname{tr} A}. \tag{3.7}$$

(d) *If $A, B \in \mathcal{I}_2$, then*

$$\det{}_2(AB) = \det{}_2 A \det{}_2 B\, \mathrm{e}^{-\operatorname{tr}(AB - A - B + 1)} =: \det{}_2 A \det{}_2 B\, \mathrm{e}^{\gamma_2(A,B)}. \tag{3.8}$$

(e) *If $A, B, C \in 1 + \mathcal{I}_2$,*

$$\omega_2(A, B) := \det{}_2 B\, \mathrm{e}^{\gamma_2(A,B)}, \tag{3.9}$$

then

$$\omega_2(A, BC) = \omega_2(AB, C)\omega_2(A, B). \tag{3.10}$$

Using Theorem 8, we prove the following lemma.

Lemma 9. *The formula*

$$(w, \lambda)s = (ws, \lambda\omega_2((1 + S), s)^{-1}), \quad s \in G^2,\ (w, \lambda) \in St_2(Gr_2(L^2_{\mathbb{R}}[0,1])) \times \mathbb{R}, \tag{3.11}$$

defines a free action from the right of G^2 on $St_2(Gr_2(L^2_{\mathbb{R}}[0,1]))$.

Proof. We first prove that (3.11) defines a right action. Let $s = 1$, then $(w, \lambda)1 = (w, \lambda\omega_2((1 + S), 1)^{-1}) = (w, \lambda)$, since $\omega_2((1 + S), 1)^{-1} = 1$. For the transitiviy, let $s, r \in G^2$, then

$$(w,\lambda)(s\circ r) = (w(s\circ r), \lambda\omega_2((1+S), sr)^{-1})$$
$$= ((ws)r, \lambda\omega_2((1+S)s, r)^{-1}\omega_2((1+S), r)^{-1})$$
$$= (ws, \lambda\omega_2((1+S), s)^{-1})r,$$

where we used (3.10). The action is free if $(w,\lambda)s = (w,\lambda) \iff s = 1$. But $(w,\lambda)s = (ws, \lambda\omega_2((1+S), s)^{-1}) = (w,\lambda)$ iff $\omega_2((1+S), s)^{-1} = 1$. The last equality exactly holds iff $s = 1$. □

Since the action of G^2 is free on the Stiefel manifold we define the smooth manifold

$$\mathrm{DET}_2 := (St_2(Gr_2(L^2_{\mathbb{R}}[0,1])) \times \mathbb{R})/G^2, \tag{3.12}$$

i.e. the determinant bundle. In order to define the tau function later on we need also the dual bundle DET_2^* of DET_2. A real analytic section on DET_2^* is a real analytic function $\psi : St_2(Gr_2(L^2_{\mathbb{R}}[0,1])) \to \mathbb{R}$ such that

$$\psi(wt) = \psi(w)\omega_2(w,t), \quad t \in G^2. \tag{3.13}$$

The next task is to determine the group acting on the determinant bundle.

Theorem 10 [4]. *For $x \in [0,1]$, x fixed, the group*

$$\hat{G}_2 := (F_2 \times \mathrm{Map}(Gr_2, \mathbb{R}^x))/N,$$

acts on DET_2, *where* $N = \{(1, q, \mu_q)\}$, $\mu_q := \alpha(1,q,w)^{-1}\omega_2((1+S), q^{-1})^{-1}$, $\mathrm{Map}(Gr_2, \mathbb{R}^x)$ *is the space of smooth functions from* Gr_2 *into* $\mathbb{R}^x$ *and*

$$\alpha : F_2 \times St_2(Gr_2(L^2_{\mathbb{R}}[0,1])) \to \mathbb{R}^x$$

is the solution of the equation

$$\frac{\alpha(g,q,wt)}{\alpha(g,q,w)} = \frac{\omega_2((1+S),t)}{\omega_2(g(1+S)q^{-1}, qtq^{-1})}, \quad (g,q) \in F_2,\ w \in St_2,\ t \in G^2.$$

If $g \in G^+_{2,0}$, the dense subgroup of G_2 where a is invertible and $c = 0$, then the sections $\phi : U \to \hat{G}_2$ form the automorphisms group of the determinant bundle.

Proof. Since G_2 acts on the Grassmannian one could think that the group action on the determinant bundle is simple given by lifting the action of the form $g(w,\lambda) = (gw,\lambda)$. The obstruction appearing is that gw is no longer an admissible basis in general. We overcome this difficulty by constructing a central extension of G^2. There is further the complication that we have regularized determinants, that is we have to regularize the determinants in order to avoid singularities. We closely follow [4] and define an action of $F_2 \times \mathrm{Map}(Gr_2, \mathbb{R}^x)$ on $St_2(Gr_2(L^2_{\mathbb{R}}[0,1])) \times \mathbb{R}$ by setting

$$(g,q,\mu)(w,\lambda) = (gwq^{-1}, \lambda\alpha(g,q,w)\mu(\pi(w))), \tag{3.14}$$

where $\pi : (St_2(Gr_2(L^2_{\mathbb{R}}[0,1]))) \to Gr_2$ is the canonical projection, $\mathrm{Map}(Gr_2, \mathbb{C}^x)$ is the space of smooth functions from Gr_2 to $\mathbb{R}^x$. The smooth function

$$\alpha : F_2 \times St_2(Gr_2(L^2_{\mathbb{R}}[0,1])) \to \mathbb{R}^x$$

is the solution of the equation

$$\frac{\alpha(g,q,wt)}{\alpha(g,q,w)} = \frac{\omega_2((1+S),t)}{\omega_2(g(1+S)q^{-1}, qtq^{-1})}, \quad (g,q) \in F_2, \; w \in St_2, \; t \in G^2. \quad (3.15)$$

The solution which takes care of the possible singularities is given by

$$\alpha(g,q,wt) = f(g,q,w)\frac{\det_2(1+S)}{\det_2(g(1+S)q^{-1})}\frac{\det_2(\frac{1}{2}(q^{-1}a(F_{11}+1)+q^{-1}cF_{21}))}{\det_2(\frac{1}{2}(F_{11}+1))}, \quad (3.16)$$

where f is an arbitrary smooth function and $F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ is the linear operator in $L^2_{\mathbb{R}}[0,1]$ which is $1(-1)$ restricted to $N \in Gr_2(L^2_{\mathbb{R}}[0,1])(N^{\perp})$, where N is spanned by the basis w (for a proof, see [4]). The composition of two group elements in (3.14) is given by

$$\begin{aligned}(g,q,\mu)(g',q',\mu') = (&gg', qq', \mu(g'N)\mu'(N)\alpha(g,q,g'wq'^{-1}) \\ &\times \alpha(g',q',w)\alpha(gg',qq',w)^{-1}).\end{aligned} \quad (3.17)$$

To prove transitivity of the action, we have

$$\begin{aligned}(g,q,\mu)(g',q',\mu') &= (gg', qq', \mu(g'N)\mu'(N)\alpha(g,q,g'wq'^{-1}) \\ &\quad \times \alpha(g',q',w)\alpha(gg',qq',w)^{-1}) \\ &= (gg'w(qq')^{-1}, \lambda\alpha(gg',qq',w)\alpha(g,q,g'wq'^{-1}) \\ &\quad \times \alpha(g',q',w)\alpha(gg',qq',w)^{-1}),\end{aligned}$$

but

$$\begin{aligned}&\alpha(gg',qq',w)\alpha(g,q,g'wq'^{-1})\alpha(g',q',w)\alpha(gg',qq',w)^{-1} \\ &= \frac{\det_2(1+S)}{\det_2(gg'(1+S)q'^{-1}q^{-1})}\frac{\det_2(\frac{1}{2}(q'^{-1}q^{-1}a(F_{11}+1)+q'^{-1}q^{-1}cF_{21})}{\det_2(\frac{1}{2}(F_{11}+1))} \\ &\quad \times \mu(g'N)\mu'(N)\alpha(g,q,g'wq'^{-1})\alpha(g',q',w)\alpha(gg',qq',w)^{-1} \\ &= \mu(g'N)\mu'(N)\alpha(g,q,g'wq'^{-1})\alpha(g'q,w).\end{aligned}$$

This implies that

$$\begin{aligned}&(gg', qq', \mu(g'N)\mu'(N)\alpha(g,q,g'wq'^{-1})\alpha(g',q',w)\alpha(gg',qq',w)^{-1}) \\ &= (g,q,\mu)((g',q',\mu')(w,\lambda)).\end{aligned}$$

This proves the transitivity. It is now easy to prove that the group action (3.14) can be lifted to an action on DET_2. However, this action is not faithful, i.e. we have to divide by the kernel which is

$$N = \{(1,q,\mu_q)\}, \quad \mu_q := \alpha(1,q,w)^{-1}\omega_2((1+S),q^{-1})^{-1}. \quad (3.18)$$

This implies that formula (3.14) defines a faithful action of the central extended group

$$\hat{G}_2 = (F_2 \times \mathrm{Map}(Gr_2, \mathbb{R}^x))/N \tag{3.19}$$

in DET_2. If $\phi : U \to \hat{G}_2$, $U \in G_2$ is a local section defined by $\phi(g) = (g, a, 1)$ then the local cocycle $\xi(g_1, g_2)$ is equal to one if g_1 and g_2 are both lower triangular, i.e. $b_1 = b_2 = 0$, and a_1 and a_2 are invertible. Then,

$$\phi(g_1)\phi(g_2) = \phi(g_1 g_2). \tag{3.20}$$

Therefore, the automorphisms group of DET_2 is $G^+_{2,0}$, the dense subgroup of G_2 where a is invertible and $c = 0$. The same is true for $G^-_{2,0}$, i.e. the subgroup of G^2 where a is invertible and $b = 0$. This proves Theorem 10. □

We now turn to the tau function. Let $\tilde{\psi}$ be a real analytic section on the dual determinant bundle DET_2^*, i.e.

$$\tilde{\psi} : St_2 \to \mathbb{R}, \quad \tilde{\psi}(wt) = \tilde{\psi}(w)\omega_2(w, t), \tag{3.21}$$

where $t \in G^2$, $w \in St_2$. A solution of Eq. (3.21) is given by $\psi(w) = \det_2 w$, which we call the canonical real analytic section. How does the other sections look like? To answer this question let $v : E \to N_q$ be the linear isomorphism which maps the ith basis element of E into the ith basis vector of N_q. If $\{U_i\}$ is an admissible basis of N_q then v has the following matrix form:

$$v = \begin{pmatrix} pr_+ \\ pr_- \end{pmatrix} \quad \text{with } pr_- \text{ Hilbert–Schmidt}, \tag{3.22}$$

that is v is a $\mathbb{Z} \times \mathbb{N}$ matrix with the column labeling the different U_i and the row labeling the coordinates in the standard basis of $L^2_{\mathbb{R}}[0, 1]$. Let $S \in \mathcal{S}$ be a fixed set and v_S the submatrix obtained from v by choosing the rows labeled by S (see the discussion after Lemma 3 for the meaning of the set $\mathcal{S}$). Then

$$\phi_S = \det_2 v_S \mathrm{e}^{\beta_2(pr_2) - \beta_2(v_S)}, \tag{3.23}$$

are the *no longer real analytic sections* of DET_2^*, where

$$\beta_2 = \log \frac{\det_2 A}{\det A}, \quad A \in 1 + \mathcal{I}_1. \tag{3.24}$$

Note that the traces in (3.24) are all taken of finite dimensional matrices. Another form to write the sections of the dual determinant bundle is given by using the one forms f_j, defined by $f_k(e_j) = \delta_{-k,j}$, where $\{e_j\}$ is an orthonormal basis of $L^2_{\mathbb{R}}[0, 1]$, and setting

$$\psi = \bigwedge_{j=0}^{-\infty} f_j, \qquad \det_2 pr_+ = \left(\bigwedge_{j=0}^{-\infty} f_j\right)(w_0, w_1, \ldots). \tag{3.25}$$

The tau function measures the non-equivariance of the group action of $G^-_{2,0}$ on the determinant bundle, i.e.

$$\tau_n(g) g^{-1} \rho(N) = \rho(g^{-1} N), \quad g \in G^-_{2,0}, \tag{3.26}$$

where $\rho(N) = (w, \det_2 pr_+)$ denotes a section on the dual bundle and where we assumed that the N is transverse to U, i.e. $N \oplus U = L^2_{\mathbb{R}}[0,1]$. We now calculate the explicit expression of the tau function. The left hand side of (3.26) is transformed in to

$$\begin{aligned} \text{LHS (3.26)} &= (g^{-1}, a, 1)(w, \lambda) = (g^{-1}wa^{-1}, \lambda\alpha(g,a,w)) \\ &= (g^{-1}wa^{-1}, \det_2 pr_+\alpha(g,a,w)) \\ &= (g^{-1}w, \det_2 a \det_2 pr_+\alpha(g,a,w)). \end{aligned}$$

The right hand side is written in the form

$$\begin{aligned} \text{RHS (3.26)} &= (g^{-1}w, \det_2(g^{-1}w)) = (g^{-1}w, \det_2(apr_+ + cpr_-)) \\ &= (g^{-1}w, \det_2 a \det_2 pr_+ \det_2(1 + a^{-1}cpr_-pr_+^{-1}) \\ &\quad \times \exp\{\gamma_2(a, a^{-1}cpr_-pr_+^{-1}) + \gamma_2(a^{-1}cpr_-pr_+^{-1}, pr_+)\}), \end{aligned}$$

where

$$\begin{aligned} \gamma_2(a, a^{-1}cpr_-pr_+^{-1}) &= -\text{tr}(cpr_-pr_+^{-1} - a - a^{-1}cpr_-pr_+^{-1} + 1), \\ \gamma_2(a^{-1}cpr_-pr_+^{-1}, pr_+) &= -\text{tr}(a^{-1}cpr_- - a^{-1}cpr_-pr_+^{-1} - pr_+ + 1). \end{aligned}$$

Since $\alpha(g,q,w) = \exp[-\text{tr}((1 - q^{-1}a)(pr_+ - 1))]$ in the case of $g \in G^-_{2,0}$, the function α above is equal to one. Therefore, we proved the lemma.

Lemma 11. *For $g \in G^-_{2,0}$, $N \in Gr_2(L^2_{\mathbb{R}}[0,1])$ and N transversal to the space of odd functions U, the tau function is given by*

$$\tau_N(g) = \det_2(1 + B)\exp\{-\text{tr}(aB - a - B + 1) + \text{tr}(Bpr_+ - pr_+ - B + 1)\}, \tag{3.27}$$

where $B = a^{-1}cpr_-pr_+^{-1}$.

Inserting the expressions $a = 1 + pr_+\mathcal{B}$ and $c = pr_+\mathcal{B}$ the tau function reads

$$\begin{aligned} \tau_N(g) &= \det_2(1 + (1 + pr_+\mathcal{B})^{-1}pr_+\mathcal{B}pr_-pr_+^{-1}) \\ &\quad \times \exp\{-\text{tr}(cpr_-pr_+^{-1} - a - a^{-1}cpr_-pr_+^{-1} + 1)\} \\ &\quad \times \exp\{\text{tr}(a^{-1}cpr_- - a^{-1}cpr_-pr_+^{-1} - pr_+ + 1)\} \\ &= \det_2(1 + pr_-pr_+ - (1 + pr_+\mathcal{B})^{-1}pr_-pr_+^{-1}) \\ &\quad \times \exp\{-\text{tr}(pr_+\mathcal{B}(pr_-pr_+^{-1} - 1) + (1 + pr_+\mathcal{B})^{-1} - 1)pr_-pr_+^{-1}\} \\ &\quad \times \exp\{\text{tr}(2pr_- - (1 + pr_+\mathcal{B})^{-1}pr_-[1 - pr_+^{-1}] - pr_-pr_+^{1})\}. \end{aligned} \tag{3.28}$$

4. Group action on the Grassmannian for arbitrary parameter x and fixed $p \in L^2_{\mathbb{R}}[0,1]$

We determine the group $\tilde{G}$ acting on the x parameter of the normal spaces of the isospectral set but leaving $q \in L^2_{\mathbb{R}}[0,1]$ fixed. Let $x, y \in [0,1]$, $y = x + h$. The main result is Theorem 12. The group element $\tilde{g} \in \tilde{G}$ we want to determine acts on two vectors of the normal space in the form

$$U(y, p(y)) = \tilde{g}(y, p(y))U(x, p(x)).$$

Then,

$$U(x+h, p(x+h)) = U(x, p(x)) + \sum_{n\geq 1} \frac{1}{n!}\left[\frac{\mathrm{d}^n U(h, p(h))}{\mathrm{d}h^k}\bigg|_{h=x} + \frac{\partial^n U(h, p(h))}{\partial h^n}\bigg|_{h=x}\right]h^n. \tag{4.1}$$

We start calculating the derivatives w.r.t. x. We set $\lambda_n = \lambda$ in the following calculations and since

$$y_2(x, \lambda, q) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x \frac{\sin\sqrt{\lambda(x-t)}x}{\sqrt{\lambda}} q(t) y_2(x, \lambda q)\,\mathrm{d}t, \tag{4.2}$$

and $|c_\lambda(t)| \leq \exp(|\mathrm{Im}\sqrt{\lambda}|t)$, we have the bound

$$|y_2' - \cos\sqrt{\lambda}x| \leq \frac{\|q\|}{|\lambda|}\exp(|\mathrm{Im}\sqrt{\lambda}|x + \|q\|\sqrt{x}), \tag{4.3}$$

hence,

$$y_2'(x, \lambda_n, q) = \cos n\pi x + \|q\|\mathrm{O}(1/n). \tag{4.4}$$

The same kind of calculations gives us for the kth-derivative

$$y_2^{(k)}(x, \lambda_n, q) = (-1)^{k-1}\sin(n\pi x + \tfrac{1}{2}k\pi) + \|q\|^k \mathrm{O}(n^{k-1}). \tag{4.5}$$

The leading order term in the kth-derivative of g_n w.r.t. x is

$$\frac{\partial_x^k y_2(x, \lambda_n, q)}{\sqrt{\dot{y}_2(1, \lambda_n) y_2'(1, \lambda_n)}}.$$

This implies the asymptotic expansion

$$\partial_x^k g_2(x, \lambda_k, q) = \sqrt{2}(\pi n)^{\frac{2k-3}{2}} \sin(\pi n x + \tfrac{1}{2}k\pi) + \mathrm{O}(n^{k-1})\|q\|^k, \tag{4.6}$$

where we used that

$$\sqrt{\dot{y}_2(1, \lambda_n) y_2'(1, \lambda_n)} = \sqrt{2}n\pi(1 + \mathrm{O}(1/n)). \tag{4.7}$$

Applying the chain rule to the second term in (4.1) we get

$$\sum_{n\geq 1}\frac{1}{n!}\Big[\frac{\mathrm{d}^n U(h,p(h))}{\mathrm{d}h^n}\Big|_{h=x}+\frac{\partial^n U(h,p(h))}{\partial h^n}\Big|_{h=x}\Big]h^n$$
$$=\sum_{n\geq 1}\sum_{k=0}^{n}\frac{1}{n!}\Big[\frac{\mathrm{d}^n U}{\mathrm{d}h^n}\Big|_{h=x}+\sum_{\mathcal{P}(n)}\frac{n!}{\prod_{j=1}^{k}\alpha_j}\frac{\partial^m U}{\partial p^m}\prod_{j=1}^{l}\frac{1}{j!}\frac{\partial^j p}{\partial h^j}\Big|_{h=x}\Big]h^n, \tag{4.8}$$

where $\sum_{\mathcal{P}(n)}$ is the sum over all positive integers α_j, $j = 1,\ldots,k$ which satisfy the equations $\alpha_1+\alpha_2+\cdots+l\alpha_k = n$ and $m=\sum_{i=1}^{k}\alpha_i$. Since

$$\partial^a U=\sum_{s=0}^{\infty}\eta_s\partial^a(g_s^2)=\sum_{s=0}^{\infty}\sum_{k=0}^{a}\eta_s\binom{a}{k}g^{(a)}g^{(a-k)},$$

we get

$$\sum_{n\geq 1}\frac{1}{n!}\Big[\frac{\mathrm{d}^n U}{\mathrm{d}h^n}+\sum_{\mathcal{P}(n)}\frac{n!}{\prod_{j=1}^{k}\alpha_j}\frac{\partial^m U}{\partial p^m}\prod_{j=1}^{l}\frac{1}{j!}\frac{\partial^j p}{\partial h^j}\Big]h^n$$
$$=\sum_{n\geq 1}\frac{1}{n!}\Bigg(\sum_{\sigma_1}\eta_{\sigma_1}\sum_{t=0}^{n}\binom{n}{t}\frac{\mathrm{d}^t g_{\sigma_1}}{\mathrm{d}h^t}\frac{\mathrm{d}^{n-t}g_{\sigma_1}}{\mathrm{d}h^{n-t}}$$
$$+\sum_{\sigma_2}\eta_{\sigma_2}\sum_{\mathcal{P}(n)}\sum_{r=0}^{m}\binom{m}{r}\frac{n!}{\prod_{j=1}^{k}\alpha_j}\frac{\partial^r g_{\sigma_2}}{\partial p^r}\frac{\partial^{m-r}g_{\sigma_2}}{\partial p^{m-r}}\prod_{j=1}^{l}\frac{1}{j!}\frac{\partial^j p}{\partial h^j}\Bigg)h^n. \tag{4.9}$$

Using the formulae

$$\mathrm{d}_h^m g_s(h,p(h),\lambda_s)=\sqrt{2}(\pi s)^{m-\frac{1}{2}s}\sin(\pi s x+\tfrac{1}{2}m\pi)+\mathrm{O}(s^{m-1})\|p\|^m,$$
$$\partial_p^l g_s(h,p(h),\lambda_s)(d,d,\ldots,d)=x^{\frac{1}{2}l}P_l(d,d,\ldots,d;p)\mathrm{O}(1/s), \tag{4.10}$$

and the trigonometric formula expressing the product of two sine functions as a sum of cosine functions, we get

$$\sum_{n\geq 1}\frac{1}{n!}\Bigg(\sum_{\sigma_1}\sum_{t=0}^{n}\Bigg[\binom{n}{t}\eta_{\sigma_1}2(\sigma_1\pi)^{n-\sigma_1}\sin\Big(2\pi\sigma_1 x+\frac{t\pi}{2}\Big)\sin\Big(2\pi\sigma_1 x+\frac{(n-t)\pi}{2}\Big)$$
$$+\|p\|^n\mathrm{O}(\sigma_1^{n-2})\Bigg]+\sum_{\sigma_2}\eta_{\sigma_2}\sum_{\mathcal{P}(n)}\frac{n!}{\prod_{j=1}^{k}\alpha_j}\sum_{r=0}^{m}\binom{m}{r}x^{\frac{1}{2}m}P_r(d,d,\ldots,d;p)$$
$$\times P_{m-r}(d,d,\ldots,d;p)\mathrm{O}\Big(\frac{1}{\sigma_2^2}\Big)\prod_{j=1}^{l}\frac{1}{j!}\frac{\partial^j p}{\partial h^j}\Bigg)h^n$$
$$=\sum_{n\geq 1}\frac{1}{n!}\Bigg(\sum_{\sigma_1}\sum_{t=0}^{n}\Bigg[\binom{n}{t}\eta_{\sigma_1}2(\sigma_1\pi)^{n-\sigma_1}\sin\Big(2\pi\sigma_1 x+\frac{t\pi}{2}\Big)$$
$$\times\sin\Big(2\pi\sigma_1 x+\frac{(n-t)\pi}{2}\Big)+\|p\|^n\mathrm{O}(\sigma_1^{n-2})\Bigg]$$
$$+\sum_{\sigma_2}\eta_{\sigma_2}\mathrm{O}\Big(\frac{1}{\sigma_2^2}\Big)F(x,\partial_h^j p)\Bigg)h^n, \tag{4.11}$$

where we defined

$$F(x,\partial_h^j p) := \sum_{\mathcal{P}(n)} 2^m \frac{n!}{\prod_{j=1}^k \alpha_j} \sum_{r=0}^{m} \binom{m}{r} P_r(d,d,\ldots,d;\,p) P_{m-r}(d,d,\ldots,d;\,p) \times \prod_{j=1}^{l} \frac{1}{j!}\frac{\partial^j p}{\partial h^j}. \tag{4.12}$$

Let $e(n)$, $n \in \mathbb{N}$, be the function which is one if n is even and zero else and $\mathrm{o}(n)$ the function which is one if n is odd and zero else. Then we can rewrite the product of the two sine functions in (4.10) in the form

$$\begin{aligned}
&\sin\Big(2\pi\sigma_1 x + \frac{t\pi}{2}\Big)\sin\Big(2\pi\sigma_1 x + \frac{(n-t)\pi}{2}\Big) \\
&= \frac{1}{2}n \prod_{k=1}^{\frac{n-2}{2}} \Big(1 - \frac{1}{\sin^2(\frac{k\pi}{n})}\Big)\mathrm{o}(n)\sin 2\pi\sigma_1 x \\
&+ \frac{1}{2}\prod_{k=1}^{\frac{n-1}{2}}\Big(1 - \frac{1}{\sin^2\big(\frac{(2k-1)\pi}{n}\big)}\Big)(\cos 2\pi\sigma_1 x + (-1)^t)e(n).
\end{aligned} \tag{4.13}$$

Using that $\sum_{t=0}^{n}\binom{n}{t} = 2^n$, $\sum_{t=0}^{n}(-1)t\binom{n}{t} = (-1)^n n!$ and (4.12), we write (4.10) in the form

$$\begin{aligned}
(4.10) = \sum_{n\geq 1}\frac{1}{n!}\Bigg(&\sum_{\sigma_1}\eta_{\sigma_1} 2\Bigg[(2^n(\sigma_1\pi)^{n-\sigma_1}\frac{1}{2}n\prod_{k=1}^{\frac{n-2}{2}}\Big(1 - \frac{1}{\sin^2(\frac{k\pi}{n})}\Big)\mathrm{o}(n)\sin 2\pi\sigma_1 x \\
&+ (2^n(\sigma_1\pi)^{n-\sigma_1}\frac{1}{2}\prod_{k=1}^{\frac{n-1}{2}}\Big(1 - \frac{1}{\sin^2\big(\frac{(2k-1)\pi}{n}\big)}\Big)(2^n\cos 2\pi\sigma_1 x + (-1)^n n!)e(n) \\
&+ 2^n\|p\|^n \mathrm{O}(\sigma_1^{n-2})\Bigg] + \sum_{\sigma_2}\eta_{\sigma_2}\mathrm{O}\Big(\frac{1}{\sigma_2^2}\Big)F(x,\partial_h^j p)\Bigg)h^n.
\end{aligned} \tag{4.14}$$

We choose now the appropriate space for η_{σ_i}, $i = 1, 2$ such that (4.13) is well defined. The first term we bound is

$$\begin{aligned}
&\Bigg|\sum_{n\geq 1}\frac{1}{n!}h^n\sum_{\sigma_1}\eta_{\sigma_1}(n(\sigma_1\pi)^{n-\sigma_1}\Bigg(2^n n\prod_{k=1}^{\frac{n-2}{2}}\Big(1 - \frac{1}{\sin^2(\frac{k\pi}{n})}\Big)\mathrm{o}(n)\sin 2\pi\sigma_1 x\Bigg)\Bigg| \\
&\leq \Bigg|\sum_{n\geq 1}\frac{1}{n!}h^n(\sigma_1\pi)^{n-\sigma_1}2^n n\prod_{k=1}^{\frac{n-2}{2}}\Big(1 - \frac{1}{\sin^2(\frac{k\pi}{n})}\Big)\mathrm{o}(n)\Bigg|\sum_{\sigma_1}|\eta_{\sigma_1}(\sigma_1\pi)^{n-\sigma_1}| \\
&=: \Bigg|\sum_{n\geq 1}\frac{1}{n!}h^n(\sigma_1\pi)^{n-\sigma_1}2^n n\prod_{k=1}^{\frac{n-2}{2}}\Big(1 - \frac{1}{\sin^2(\frac{k\pi}{n})}\Big)\mathrm{o}(n)\Bigg|\,|A(n)|.
\end{aligned}$$

Using the Hölder inequality we estimate

$$|A(n)| \le \pi^{-n} \sqrt{\sum_{\sigma_1} \left(\frac{\eta_{\sigma_1}}{\pi^{\sigma_1}}\right)^2} \sqrt{\sum_{\sigma_1} \left(\frac{{\sigma_1}^{2n}}{{\sigma_1}^{2\sigma_1}}\right)}$$
$$\le \pi^{-n} c \sqrt{\sum_{\sigma_1} \left(\frac{{\sigma_1}^{2n}}{{\sigma_1}^{2\sigma_1}}\right)}, \quad \text{for } \eta_{\sigma_1} \in \ell^2 \le \pi^{-n} c \sqrt{\sum_{\sigma_1} \frac{{\sigma_1}^{2n}}{(\sigma_1 !)^2}}.$$

But the terms S_n, defined by

$$\sum_{\sigma_1} \frac{{\sigma_1}^n}{(\sigma_1 !)} = S_n$$

or in the equivalent way by

$$S_n = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{\mathrm{e}^x} \Big|_{x=0}$$

are bounded by $S_n \le n!n/([n/2])!$, where $[n/2]$ is for n odd equal to the integer N: $\frac{1}{2}n - 1 < N < \frac{1}{2}n$. Hence we get for the first term

$$\Big| \sum_{n \ge 1} \frac{1}{n!} h^n (\sigma_1 \pi)^{n-\sigma_1} 2^n n \prod_{k=1}^{\frac{n-2}{2}} \left(1 - \frac{1}{\sin^2(k\pi/n)}\right) \mathrm{o}(n) \Big| \, |A(n)|$$
$$\le \Big| \sum_{n \ge 1} h^n 2^n n^2 \mathrm{o}(n) \frac{1}{([n/2])!} q^{\frac{n-2}{2}} \Big| < \infty, \quad \text{for } \eta_{\sigma_1} \in \ell^2,$$

where we used

$$\prod_{k=1}^{\frac{n-2}{2}} \Big| \left(1 - \frac{1}{\sin^2(k\pi/n)}\right) \Big| \le \frac{\prod_{k=1}^{\frac{n-2}{2}} (\sin^2(k\pi/n) - 1)}{\prod_{k=1}^{\frac{n-2}{2}} \sin^2(k\pi/n)} \le q^{\frac{n-2}{2}},$$

with $q \in [0, 1]$.The next term we bound is

$$\Big| \sum_{n \ge 1} \frac{1}{n!} h^n 2 \sum_{\sigma_1} \eta_{\sigma_1} \|p\|^n \mathrm{O}(\sigma_1^{n-2}) \Big| \le c \sum_{n \ge 1} \frac{1}{n!} |h^n| \, \|p\|^n \sum_{\sigma_1} |\eta_{\sigma_1}| \mathrm{O}(|\sigma_1^{n-2}|),$$

which converges if $\{\eta_{\sigma_n}\}_{n \in \mathbb{N}} \in s$, where s is the discrete analogue to the Schwartz space $\mathcal{S}$. The third term we consider is

$$\Big| \sum_{n \ge 1} \frac{1}{n!} \sum_{\sigma_2} \eta_{\sigma_2} \mathrm{O}\Big(\frac{1}{\sigma_2^2}\Big) F(x, \partial_h^j p) h^n \Big|$$
$$\le \sum_{n \ge 1} \frac{1}{n!} |h^n| \sum_{\sigma_2} \Big| \eta_{\sigma_2} \mathrm{O}\Big(\frac{1}{\sigma_2^2}\Big) \Big| \sum_{\mathcal{P}(n)} 2^m \frac{n!}{\prod_{j=1}^k \alpha_j}$$
$$\times \sum_{r=0}^m \binom{m}{r} |P_r(d, d, \ldots, d; p) P_{m-r}(d, d, \ldots, d; p)| \Big| \prod_{j=1}^l \frac{1}{j!} \frac{\partial^j p}{\partial h^j} \Big|$$
$$\le \sum_{n \ge 1} \frac{1}{n!} |h^n| \sum_{\sigma_2} \Big| \eta_{\sigma_2} \mathrm{O}\Big(\frac{1}{\sigma_2^2}\Big) \Big| \sum_{\mathcal{P}(n)} 2^m \frac{n!}{\prod_{j=1}^k \alpha_j} \frac{1}{\|d\|^m} m^3 \Big| \prod_{j=1}^l \frac{1}{j!} \frac{\partial^j p}{\partial h^j} \Big|,$$

where we assumed that $\|d\| > 1, |\langle d, p \rangle|, |\langle d, d \rangle| < \|d\|$. If we take $p(h) \in \mathcal{S}$, then

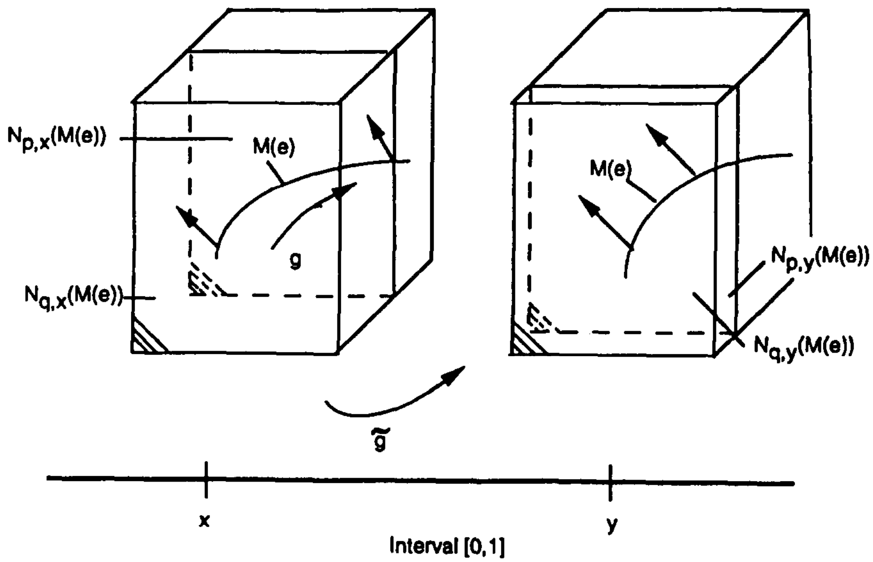

Fig. 4.

$$\leq \sum_{n\geq 1} \frac{1}{n!} |h^n| \sum_{\sigma_2} \left| \eta_{\sigma_2} \mathrm{O}\left(\frac{1}{\sigma_2^2}\right) \right| \sum_{\mathcal{P}(n)} 2^m \frac{n!}{\prod_{j=1}^k \alpha_j} \frac{1}{\|d\|^m} m^3 \left| \prod_{j=1}^{l} \frac{1}{j!} \frac{\partial^j p}{\partial h^j} \right|$$
$$\leq c \sum_{n\geq 1} \frac{1}{n!} |h^n| \sum_{\sigma_2} \left| \eta_{\sigma_2} \mathrm{O}\left(\frac{1}{\sigma_2^2}\right) \right| \leq \infty.$$

We proved the following lemma.

Theorem 12. *For fixed $p(x) \in M(e)$, the function $\tilde{g}$, defined in* (4.1), *acting on a normal vector $U(x, p(x))$ transforms it into the normal vector given in* (4.10). *The transformation $\tilde{g}$ is well defined if the sequence $\{\eta_{\sigma_1}\}_{n\in\mathbb{N}}$ is an element of the discrete Schwarz space s, $\{\eta_{\sigma_1}\}_{n\in\mathbb{N}} \in \ell^2$ and $p(x) \in \mathcal{S}[0, 1]$.*

5. Conclusion, Fock bundle

We saw in the previous sections that there are two groups GL_2 and $\tilde{G}$ acting on the normal spaces, i.e. on the Grassmannian Gr_2, and that there exists a central extension of G^2 which acts on the determinant bundle constructed over the Grassmannian. Fig. 4 shows the geometric situation.

The question is, whether there exists a bundle $\mathcal{F}$ and if there exists a group M which has G^2 and an extension of $\tilde{G}$ as subgroups and which acts on the bundle $\mathcal{F}$ as an automorphism group. The way to find an answer to this question is to consider a situation where a similar construction has to be carried out and then to see how this construction can be used in our case.

The example we discuss is 3 + 1 dimensional Dirac–Yang–Mills theory. We follow Mickelsson [4] where the details can be found. The Fock space, that is the space of holomorphic sections on a dual determinant bundle, is described by a single Dirac operator. This operator contains a potential A and if a physical theory wants to describe any interaction one has to consider not only one potential but all of them which describe a Dirac operator. This family of operators hence describes a family of Fock spaces. This implies that the splitting of the underlying Hilbert space $H = H_+(A) \oplus H_-(A)$, which one uses to construct for example the Grassmannian and determinant bundle, depends on the parameter A. To be more explicit we consider the $3+1$ theory. Then H_+ is the direct sum of eigenspaces of the Dirac operators with potential A and positive energy. Furthermore, let M be a 3 dimensional compact spin manifold, G a compact Lie group, G the Lie algebra of G and $\mathcal{A}$ the vector space of all G-valued one forms. Then the map $\mathcal{A} \to Gr_2$, $A \to H_+(A)$, is not continuous at the points in the space $\mathcal{A}$ where the eigenvalues of the Dirac Hamiltonian are zero. To overcome this difficulty in constructing the Fock bundle $\mathcal{F}_A$, $A \in \mathcal{A}$, one either constructs the projective space of complex lines of $\mathcal{F}_A$ or one considers the Fock space parametrized by the elements of Gr_2 and not by those of $\mathcal{A}$. We discuss the second approach.

If $F \in Gr_2$, the Fock space $\mathcal{F}_F$ w.r.t. $H = F \oplus F^\top$ is defined by the choice of a basis $f = (f_1, f_2, \ldots)$ in F which is admissible to the basis $e = (e-1, e_2, \ldots)$ of H_+, that is

$$pr_+(f_n) = \sum_{j=1}^{\infty} f_{jn} e_j, \qquad pr_F(w_n) = \sum_{j=1}^{\infty} w_{jn}^{(f)} f_j,$$

where $(f_{ij})_{i,j \in \mathbb{N}}$ is an operator of the form $1 + \mathcal{I}_2$, $w \in St_2$ and $w^{(f)}$ is the matrix appearing in the sections, i.e.

$$\psi(wt) = \psi(w)\omega_2(w^{(f)}, t), \quad t \in G^2.$$

Since ψ depends on f we write $\psi(w, f)$.

Theorem 13. *The function $\psi : St_2 \times St_2 \to \mathbb{C}$ which satisfies the equations:*

$$\psi(wt, f) = \psi(w, f)\omega_2(w^{(f)}, t), \quad t \in G^2,$$
$$\psi(w, ft) = \psi(w, f)\frac{\det_2(w^{(f)}t)}{\det_2 w^{(f)}}$$

is a section on the vector bundle $\mathcal{F}'$ over Gr_2, where $\mathcal{F}' = \mathrm{DET}_2^* \otimes \mathcal{B}$ and $\mathcal{B}$ is the trivial Fock bundle over Gr_2 with fiber $\mathcal{F}_{H_+}$.

The vacuum vector is $\psi_0 = \det_2(f^* w)$ and there exist groups G_l, G_r and G_d which act in the following way on the sections:

$$g_l(g, q, \lambda)\psi(w, f) = \alpha_l(g, q, w, f)\psi(g^{-1}wq, f),$$
$$g_r(g, q, \lambda)\psi(w, f) = \alpha_r(g, q, w, f)\psi(w, g^* fq),$$
$$g_d(g)\psi(w, f) = \psi(g^{-1}wq, g^* fq)$$

with $g \in G^2, q \in GL(H_+)$ and the functions α are determined in the same way as in Theorem 10. The interpretation is the following one: f are the background fields and G_l is the group of gauge transformations in the fermionic Fock space $\Gamma(\mathrm{DET}_2^*)$. The group G_r is the action of the gauge transformations on the vector potentials and G_d is the symmetry group of the coupled Dirac–Yang–Mills system. Let $\mathcal{F}_{\mathrm{hol}}$ be the subbundle of $\mathcal{F}'$ such that the fiber of $\mathcal{F}_{\mathrm{hol}}$ at $F \in Gr_2$ consists of all holomorphic sections of $\mathrm{DET}_2^*(F)$, that is, a section ψ of $\mathcal{F}_{\mathrm{hol}}$ is given by

$$\psi(w, f) = \lambda(F)\psi_S(w, f),$$

with $\lambda : Gr_2 \to \mathbb{C}$ a smooth function and $S : H \to H$ an operator which is equal to the identity operator plus a finite rank one. We know that G^2 does not act in the space of holomorphic section $\Gamma_{\mathrm{hol}}(\mathrm{DET}_2^*)$ but the central extension of the group $G^{2/3}$ does.

We compare this construction with our case. Let $U = (U_1, U_2, \ldots)$, $V = (V_1, V_2, \ldots)$ be two admissible bases. Furthermore, the Fock space $\mathcal{F}_{N_q(M(e))}$ w.r.t. the splitting $L^2_{\mathbb{R}}[0, 1] = N_q(M(e)) \oplus N_q(M(e))^{\top}$ and the projection $pr_+ : N_q(M(e)) \to E$ is given. The groups which act on the sections of the Fock bundle are G^2 and $\tilde{G}$, that is

$$\psi(Ug, V) = \psi(U, V)\omega(U^{(V)}, g), \quad g \in G^2,$$
$$\psi(U, V\tilde{g}) = \psi(U, V)\frac{\det_2(U^{(V)}\tilde{g})}{\det_2 U^{(V)}}, \quad \tilde{g} \in \tilde{G}.$$

An element $\tilde{g}$ has the form $\tilde{g} = (\mu, \mu 1)$, $\mu \in \mathbb{R}$, and it acts on a normal vector by $V(x, q(x))\tilde{g} = V(\mu x, g(\mu x))$.

Appendix A

Proof of Lemma 6. The normalized eigenfunctions are $g_n(x, \lambda_n) = y_2(x, \lambda_n)/\sqrt{\dot{y}_2(1, \lambda_n)y_2'(1, \lambda_n)}$, where "$\cdot$" denotes differentiation w.r.t. λ and "$'$" w.r.t. x. Furthermore,

$$y_2(x, \lambda_n) = s_{\lambda_n}(x) + \sum_{m\geq 1} S_m(x, \lambda_n), \quad s_{\lambda_n}(x) := \frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}},$$
$$S_m(x, \lambda_n, q(x)) = \int\limits_{0\leq t_1\leq\cdots\leq t_{m+1}=x} s_{\lambda_n}(t_1)\prod_{i=1}^{m}(s_{\lambda_n}(t_{i+1} - t_i)q(t_i))\,\mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_m.$$

The derivative of g_n w.r.t. q is given by

$$\partial_q g_n := \frac{\partial_q y_2(x, \lambda_n)}{\sqrt{\dot{y}_2(1, \lambda_n)y_2'(1, \lambda_n)}} - \frac{y_2(x, \lambda_n)\partial_h(\dot{y}_2(1, \lambda_n)y_2'(1, \lambda_n))}{\left(\sqrt{\dot{y}_2(1, \lambda_n)y_2'(1, \lambda_n)}\right)^3}.$$

First we write out S_m more explicitly, i.e.

$$S_m(x,\lambda_n,q(x)) = \int_0^x s_{\lambda_n}(x-t_m)q(t_m)\left(\int_0^{t_m} s_{\lambda_n}(t_m-t_{m-1})q(t_{m-1})\right.$$
$$\times\left(\int_0^{t_{m-1}} s_{\lambda_n}(t_{m-1}-t_{m-2})q(t_{m-2})\right.$$
$$\left.\left.\times\left(\dots\left(\int_0^{t_2} s_{\lambda_n}(t_1)s_{\lambda_n}(t_2-t_1)q(t_1)\,\mathrm{d}t_1\right)\right)\times\right)\mathrm{d}t_{m-1}\right)\mathrm{d}t_m,$$

then

$$\partial_h S_m(x,\lambda_n,h(x)) = \int_0^x s_{\lambda_n}(x-t_m)[f(h(t_m)) + h(t_m)\partial_h f(h(t_m))]\,\mathrm{d}t_m.$$

Since $|s_\lambda(x)| = |\int_0^x c_\lambda(t)\,\mathrm{d}t| \leq \exp[|\mathrm{Im}\sqrt{\lambda}|\,x]$, where we used

$$c_\lambda := \cos\lambda x = \tfrac{1}{2}|\exp[\mathrm{i}\sqrt{\lambda}x] + \exp[-\mathrm{i}\sqrt{\lambda}x]|,$$

we estimate S_m by

$$|S_m(x,\lambda_n,h(x)| \leq \int_{0\leq t_1\leq\cdots\leq t_{m+1}=x} |s_{\lambda_n}(t_1)|\prod_{i=2}^{m}|(s_{\lambda_n}(t_{i+1}-t_i)h(t_i))|\,\mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_m$$
$$= \frac{\exp[|\mathrm{Im}\sqrt{\lambda}|x]}{|\lambda_n|}\frac{1}{m!}\left(\int_{[0,1]}|h(t)|\mathrm{d}t\right)^m$$
$$\leq \frac{\exp[|\mathrm{Im}\sqrt{\lambda}|x]}{|\lambda_n|}\frac{1}{m!}(\|h\|\sqrt{x})^m.$$

Therefore,

$$|\partial_h S_m(x,\lambda_n,h(x)|$$
$$\leq \int_0^x s_{\lambda_n}(x-t_m)[f(h(t_m)) + h(t_m)\partial_h f(h(t_m))]\,\mathrm{d}t_m$$
$$\left|\int_0^x s_{\lambda_n}(x-t_m)h(t_m)\int_0^{t_m} s_{\lambda_n}(t_{m-1})s_{\lambda_n}(t_m-t_{m-1})h(t_{m-1})\partial hf(h(t_{m-1}))\,\mathrm{d}t_{m-1}\,\mathrm{d}t_m\right|$$
$$\leq m\frac{\exp[|\mathrm{Im}\sqrt{\lambda}|x]}{|\lambda_n|}\sqrt{x}\frac{1}{(m-1)!}(\|h\|\sqrt{x})^{m-1}.$$

This implies the bound

$$|\partial_h y_2| \le \frac{\exp[|\mathrm{Im}\sqrt{\lambda}|x]}{|\lambda_n|} \sum_{m=1}^{\infty} m\sqrt{x}\frac{1}{(m-1)!}(\|h\|\sqrt{x})^{m-1}$$
$$\le \frac{\exp[|\mathrm{Im}\sqrt{\lambda}|x]}{|\lambda_n|} \mathrm{e}^{(\|h\|\sqrt{x})}\sqrt{x}\frac{\langle h, \bullet\rangle}{\|h\|}.$$

The second term in the derivative of g_n is bounded using

$$\sqrt{\dot{y}_2(1,\lambda_n)y_2'(1,\lambda_n)} = \sqrt{2}\sqrt{\lambda_n}(1+\mathrm{O}(1/n))^{-1},$$

which implies that

$$\frac{\partial_h y_2}{\sqrt{\dot{y}_2(1,\lambda_n)y_2'(1,\lambda_n)}} = \sqrt{x}\tfrac{\langle h,\bullet\rangle}{\|h\|}\mathrm{O}(\tfrac{1}{n^2})(\sqrt{2}\sqrt{\lambda_n}(1+\mathrm{O}(\tfrac{1}{n}))) = \sqrt{x}\tfrac{\langle h,\bullet\rangle}{\|h\|}\mathrm{O}(\tfrac{1}{n})$$

Since,

$$\partial_h(\dot{y}_2(1,\lambda_n)y_2'(1,\lambda_n)) = \partial_h \int_0^1 y_2^2(t,\lambda_n)\,\mathrm{d}t = 2\int_0^1 y_2(t,\lambda_n)\partial_h y_2(t,\lambda_n)\,\mathrm{d}t$$
$$= \sqrt{x}\frac{\langle h,\bullet\rangle}{\|h\|}\mathrm{O}\Big(\frac{1}{n^2}\Big)\int_0^1 \Big(\frac{2\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}} + \mathrm{O}\Big(\frac{1}{n^2}\Big)\Big)\,\mathrm{d}t$$
$$= \sqrt{x}\frac{\langle h,\bullet\rangle}{\|h\|}\mathrm{O}\Big(\frac{1}{n^4}\Big),$$

this gives us the following expansion for the second term:

$$\frac{y_2(x)\partial_h(\dot{y}_2(1)y_2'(1))}{(\sqrt{\dot{y}_2(1)y_2'(1)})^3}$$
$$= \Big(\frac{2\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}} + \mathrm{O}\Big(\frac{1}{n^2}\Big)\Big)\sqrt{x}\frac{\langle h,\bullet\rangle}{\|h\|}\mathrm{O}\Big(\frac{1}{n^4}\Big)\Big(\sqrt{2}\sqrt{\lambda_n}\Big(1+\mathrm{O}\Big(\frac{1}{n}\Big)\Big)\Big)$$
$$= \sqrt{x}\frac{\langle h,\bullet\rangle}{\|h\|}\mathrm{O}\Big(\frac{1}{n^4}\Big).$$

The first dominating term has the expansion

$$\frac{\langle h,\bullet\rangle}{\|h\|}\sqrt{x}\mathrm{O}\Big(\frac{1}{n}\Big).$$

This proves part (a). To prove part (b) we start with

$$\partial_h^k y_2 = \sum_{m=1}^{\infty} \partial_h^k S_m(x,\lambda_n,h(x))$$

$$= \sum_{m=1}^{\infty} \int_{0 \le t_1 \le \cdots \le t_{m+1} = x} s_{\lambda_n}(t_1) m(m-1)(m-2)\cdots(m-k)$$
$$\times \prod_{i=1}^{m-k} (s_{\lambda_n}(t_{i+1} - t_i) q(t_i))\, \mathrm{d}t_1\, \mathrm{d}t_2 \cdots \mathrm{d}t_m,$$

which implies the bound

$$|\partial_h^k y_2| \le \frac{\exp[|\operatorname{Im}\sqrt{\lambda}|x]}{|\lambda_n|} \sum_{m=1}^{\infty} m(m-1)(m-2)\cdots(m-k)(\|h\|\sqrt{x})^{m-k}$$
$$= \frac{\exp[|\operatorname{Im}\sqrt{\lambda}|x]}{|\lambda_n|} \frac{\partial^k}{\partial h^k} \mathrm{e}^{(\|h\|\sqrt{x})}.$$

The derivative of $(\partial^k/\partial h^k)\mathrm{e}^{(\|h\|\sqrt{x})}$ is given by

$$\frac{\partial^k}{\partial h^k} \mathrm{e}^{(\|h\|\sqrt{x})} = \sum_{i=1}^{k} U_i(y) F^i(y),$$

where

$$F(y(x)) = \exp[y(x)] = \exp[\|h\|\sqrt{x}],$$
$$U^i = \partial_h^k y^i - \frac{i}{1!} y \partial_h^k y^{i-1} + \frac{(i-1)i}{2!} y^2 \partial_h^k y^{i-2} - \cdots + (-1)^{i-1} i y^{i-1} \partial_h^k y.$$

From this last formula the bound (2.28) is easily obtained. □

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